Renormalization group in Statistical Mechanics and Mechanics: gauge symmetries and vanishing beta functions.

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Abstract: *Two very different problems that can be studied by renormalization group methods are discussed with the aim of showing the conceptual unity that renormalization group has introduced in some areas of theoretical Physics. The two problems are: the ground state theory of a one dimensional quantum Fermi liquid and the existence of quasi periodic motions in classical mechanical systems close to integrable ones. I summarize here the main ideas and show that the two treatments, although completely independent of each other, are strikingly similar.*

§1. Introduction.

There are few cases in which a renormalization group analysis can be performed in full detail and without approximations. The best known case is the *hierarchical model* theory of Wilson, [Wi70], [WK74]. Other examples are the (Euclidean) φ^4 quantum field theories in two and three space–time dimensions, [Wi73] (for an analysis in the spirit of what follows see [Ga85] or [BG95]), and the universality of critical points [WF72]. In all such examples there is a basic difficulty to overcome: namely the samples of the fields can be unboundedly large: this does not destroy the method because such large values have extremely small probability, [Ga85]. The necessity of a different treatment of the large and the small field values hides, to some extent, the intrinsic simplicity and elegance of the approach: unnecessarily so as the end result is that one can essentially ignore (to the extent that it is not even mentioned in most application oriented discussions) the large field values and treat the renormalization problem perturbatively, as if the large fields were not possible.

Here I shall discuss two (non trivial) problems in which the large field difficulties are not at all present, and the theory leads to a convergent perturbative solution of the problem (unlike the the above mentioned classical cases, in which the perturbation expansion cannot be analytic in the perturbation parameter).
The problems are:

(1) the theory of the ground state of a system of (spinless, for simplicity) fermions in 1-dimension, [BGPS],[BG90],[BM95];
(2) the theory of KAM tori in classical mechanics, [El96], [Ga95], [BGGM], [GGM95].

The two problems will be treated *independently*, for completeness, although it will appear that they are closely related. Since the discussion of problem (1) is quite technical we summarize it at the end (in §3) in a form that shows the generality of the method that will then be applied to the problem (2) in §4.
The analysis of the above examples suggests methods to study and solve several problems in the theory of rapidly perturbed quasi periodic unstable motion, [Ga95], [GGM99]: but for brevity we shall only refer to the literature for such applications.

§2. Fermi systems in one dimension.

The Hamiltonian for a system of N *spinless* fermions at $\underline{x}_1, \dots, \underline{x}_N$ enclosed in a box (actually an interval) of size V is:

2.1
$$H = \sum_{i=1}^{N} \big(\frac{1}{2m} \Delta_{\underline{x}_i} - \mu \big) + 2\lambda \sum_{i<j} v(\underline{x}_i - \underline{x}_j) - \sum_{i=1}^{N} \nu \tag{2.1}$$

where μ is the chemical potential, v is a smooth interaction pair potential, λ is the strength of the coupling; ν is a correction to the chemical potential that vanishes for $\lambda = 0$ and that has to be adjusted as a function of λ; it is introduced in order that the Fermi momentum stays λ–independent and equal, therefore, to $p_F = (2m\mu)^{\frac{1}{2}}$. It is in fact convenient to develop the theory at fixed Fermi momentum because the latter has a more direct physical meaning than the chemical potential as it marks the location of important singukarities of the functions that describe the theory. The parameter m is the mass of the particles in absence of interaction.

It is well known, [LW60], that the ground state of the above Hamiltonian is described by the Schwinger functions of a fermionic theory whose fields will be denoted $\psi^{\pm}_x$. For instance the *occupation number* function $n_{\underline{k}}$ which, in absence of interaction, is the simple characteristic function $n_{\underline{k}} = 1$ if $|\underline{k}| < p_F$ and $n_{\underline{k}} = 0$ if $|\underline{k}| > p_F$ is, in general, the Fourier transform of $S(\underline{x}, t)$ with $x = (\underline{x}, t) = (\underline{x}_1 - \underline{x}_2, t_1 - t_2)$ and $t = 0^+$:

$$S(\underline{x}) = S(\underline{x}_1, t_1; \underline{x}_2, t_2)\big|_{t_1 = t_2^+} = \lim_{\substack{\beta \to \infty \\ V \to \infty}} \frac{\mathrm{Tr}\, e^{-(\beta - t_1)H} \psi^+_{x_1,t_1} e^{-(t_1 - t_2)H} \psi^-_{x_2,t_2} e^{-t_2 H}}{\mathrm{Tr}\, e^{-\beta H}}\Big|_{t_1 = t_2^+} \tag{2.2}$$

2.2

Formal perturbation analysis of the 2–points Schwinger function $S(x)$ and of the n–points natural extensions $S(x_1, x_2, \dots, x_n)$ can be done and the (heuristic) theory is very simple in terms of Feynman graphs.

The n-points Schwinger function is expressed as a power series in the couplings λ, ν $\sum_{p=0}^{\infty} \lambda^p \nu^q S^{(p,q)}(x_1, x_2, \dots, x_n)$ with the coefficients $S^{(p,q)}$ computed by considering the (connected) Feynman graphs composed by linking together in all possible ways the following basic "*graph elements*"

(1) p "internal 4–lines graph elements" (also called "coupling graphs") and q "internal 2–lines graph elements" (or "chemical potential vertices") of the form:

Fig. 1: The two basic building blocks ("graph elements") of the the Feynman graphs for the description of the ground state: the first represents the potential term ($2\lambda v$) in (2.1)) and the second the chemical potential term (ν).

where the incoming or outgoing arrows represent ψ^-_x or ψ^+_x, respectively, and

(2) n single lines attached to "external" vertices $\underline{x}_j$: the first half of which oriented towards the vertex x and the other half of them oriented away from it:

Fig. 2: Graphical representation of the "external" lines and vertices in Feynman graphs.

The graphs are formed by *contracting* (*i.e.* joining) together lines with equal orientation. The lines emerging from different nodes are regarded as distinct: we can imagine that

each line carries a label distinguishing it from any other, *e.g.* the lines are thought to be numbered from 1 to 4 or from 1 to 2, depending on the structure of the graph element to which they belong. So that there are many graphs giving the same contributions.

Each graph is assigned a *value* which is $\pm(p!q!)^{-1}\lambda^p\nu^q$ times a product of *propagators*, one per line. The propagator for a line joining x_1 to x_2 is, if $x_1 = (\underline{x}_1, t_1)$, $x_2 = (\underline{x}_2, t_2)$:

x_1 ●⟶● x_2

$$g(x_1 - x_2) = \frac{1}{(2\pi)^2}\int \frac{e^{-i(k_0(t_1-t_2)+\underline{k}(\underline{x}_1-\underline{x}_2))}}{-ik_0 + (\underline{k}^2 - p_F^2)/2m}\, dk_0 d\underline{k} \tag{2.3}$$

A *wavy line*, see Fig. 1, joining x_1 with x_2 is also given a propagator

$$\tilde{g}(x_2 - x_1) = v(\underline{x}_2 - \underline{x}_1)\,\delta(t_2 - t_1) \tag{2.4}$$

associated with the "potential". However the wavy lines are necessarily internal as the can only arise from the first graph element in Fig. 1.

The $p + q$ *internal* node labels $(\underline{x}, t)$ must be integrated over the volume occupied by the system (*i.e.* the whole space–time when $V, \beta \to \infty$): the result will be called the "integrated value" of the graph or simply, if not ambiguous, the graph value.

Since the value of a graph has to be integrated over the labels $x = (\underline{x}, t)$ of the internal nodes we shall often consider also the value of a graph ϑ *without the propagators corresponding to the external lines* but integrated with respect to the positions of all nodes that are not attached to external lines and we call it the "*kernel*" of the graph ϑ: the value of a graph ϑ will often be denoted as $\mathrm{Val}\,\vartheta$ and the kernel by K_ϑ. Note that the kernel of a graph depends on less variables: in particular it depends only on the positions of the internal nodes; it also depends on the labels $\underline{\omega}$ of the branches external to them through which they are connected to the external vertices.

Introducing the notion of kernel is useful because it makes natural to collect together values of graphs which contain subgraphs with the same number of lines exiting them, *i.e.* whose kernels have the same number of variables.

The function $\lambda^p\nu^q S^{(p,q)}$ is given by the sum of the values of all Feynman graphs with p vertices of the first type in Fig. 1 and q of the second type in Fig. 1 and, of course, n external vertices, *integrated* over the internal vertices positions. As an example consider the following contribution to $S^{(4,2)}$:

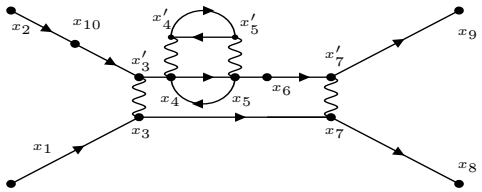

Fig. 3: An example of a Feynman graph: in spite of its involved structure it is far simpler than its numerical expression, see (2.5). A systematic consideration of graphs as "short cuts" for formulae permits us to visualize more easily various quantities and makes it possible to recognize cancellations due to symmetries.

The value of the above graph can be easily written in formulae: apart from a global sign that has to be computed by a careful examination of the order in which the x_j-labels are written it is

$$\begin{aligned} S^{(4,2)}_\vartheta(x_1, x_2, x_8, x_9) = \pm\frac{1}{4!\,2!}\int\ & g(x_{10} - x_2)\, g(x_3 - x_1)\, g(x'_3 - x_{10})\cdot \\ & \cdot g(x_4 - x'_3)\, g(x_5 - x_4)\, g(x_4 - x_5)\, g(x'_5 - x'_4)\, g(x'_4 - x'_5)\, g(x_6 - x_5)\cdot \\ & \cdot g(x_4 - x_5)\, g(x_5 - x_4)\, g(x'_7 - x_6)\, g(x_9 - x'_7)\, g(x_8 - x_7)\, \delta(t_4 - t'_4)\cdot \\ & \cdot \delta(t_5 - t'_5)\, v(\underline{x}'_3 - \underline{x}_3)\, \delta(t'_3 - t_3)\, v(\underline{x}'_7 - \underline{x}_7)\, \delta(t'_7 - t_7)\cdot \\ & \cdot v(\underline{x}'_4 - \underline{x}_4)\, v(\underline{x}'_5 - \underline{x}_5)\, dx_3\, dx'_3\, dx_4\, dx'_4\, dx_5\, dx'_5\, dx_6\, dx_7\, dx'_7 dx_{10} \end{aligned} \tag{2.5}$$

which is easily derived from the figure. And one hardly sees how this formula could be useful, particularly if one thinks that this is but *one* of a large number of possibilities that arise in evaluating S: not to mention what we shall get when looking at higher orders, *i.e.* at $S^{(p,q)}$ when p is a bit larger than 2.

Many (in fact most) of the integrals over the node variables x_v will, however, *diverge*. This is a "trivial" divergence due to the fact that interaction tends to change the value of the chemical potential. The chemical potential is related (or can be related) to the Fermi field propagator singularities, and the chemical potential is changed (or may be changed) by the interaction: the divergences are due to the naiveté of the attempt at expanding the functions S in a power series involving functions with singularities located "at the wrong places".

The divergences disappear if the (so far free) parameter ν is chosen to depend on λ as:

2.6
$$\nu = \sum_{k=1}^{\infty} \nu_k \lambda^k \tag{2.6}$$

with the coefficients ν_k suitably defined so that the resulting power series in the single parameter λ has coefficients free of divergences, [LW60].

This leads to a power series in just one parameter λ and the "only" problem left is therefore that of the convergence of the expansion of the Schwinger functions in powers of λ. This is non trivial because naive estimates of the sum of all graphs contributing to a given order p yield bounds that grow like $p!$, thus giving a vanishing estimate for the radius of convergence.

The idea is that there are cancellations between the values of the various graphs contributing to a given order in the power series for the Schwinger functions: and that such cancellations can be best exhibited by further breaking up the values of the graphs and by again combining them conveniently.

The "renormalization group method" can be seen in different ways: here I am proposing to see it as a resummation method for (possibly divergent) power series.

Keeping the original power series in λ, ν, *i.e.* postponing the choice of ν as a function of λ, one checks the elementary fact that the propagator $g(x)$ can be written, setting $k = (k_0, \underline{k}) \in R^2$, also as:

2.7
$$\begin{aligned} g(x) &= \sum_{h=-\infty}^{1} \sum_{\underline{\omega}=\pm 1} e^{i\,\underline{\omega}\,p_F\,\underline{x}} 2^h g^{(h)}_{\underline{\omega}}(2^h p_F x) \\ \hat{g}^{(h)}_{\underline{\omega}}(k) &= \frac{\chi^{(h)}(k)}{-ik_0 + \underline{\omega}\,\underline{k}} + \text{"neglegible corrections"} \end{aligned} \tag{2.7}$$

where $\chi^{(1)}(k)$ is a function increasing from 0 to 1 between $\frac{1}{2}p_F$ and p_F, while the functions $\chi^{(h)}(k)$ are the same function scaled to have support in $2^{h-2}p_F < |k| < 2^h p_F$. This means that for $h \le 0$ it is $\chi^{(h)}(k) = \chi(2^{-h}kp_F^{-1})$. The simplest choice is to take $\chi^{(1)}(k)$ to be the characteristic function of $z \equiv p_F^{-1}|k| > 1$ and $\chi(z)$ to be the characteristic function of the interval $[\frac{1}{2}, 1]$:

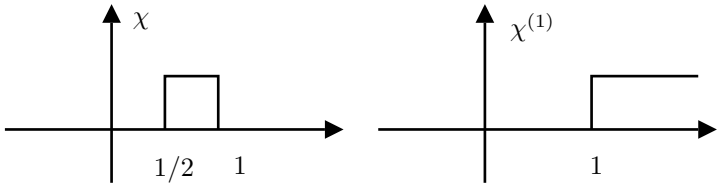

Fig. 4: A (non smooth) scaling (by a factor of 2) decomposition of unity.

so that

2.8
$$\sum_{h=-\infty}^{1} \chi^{(h)}(k) \equiv 1 \tag{2.8}$$

To avoid technical problems it would be convenient to smoothen the discontinuities in Fig. 4 of χ and $\chi^{(1)}$ turning them into C^∞–functions which in a small vicinity of the jump increase from 0 to 1 or decrease from 1 to 0, this is possible while *still keeping the scaling decomposition* (2.8) (*i.e.* with $\chi^{(h)}(k) \equiv \chi(k)$). However the formalism that this smoothing would require is rather havy and hides the stucture of the approach; therefore we shall continue with the decomposition of unity in (2.8) with the sharply discontinuous functions in Fig. 4, warning (*c.f.r.* footnote [2] below) the reader when this should cause a problem.

The "negligible terms" in (2.7) are terms of a similar form but which are smaller by a factor 2^h at least: their presence does not alter the analysis other than notationally. *We shall henceforth set them equal to* 0 *because taking them into account only introduces notational complications.*

The above is an *infrared scale decomposition* of the propagator $g(x)$: in fact the propagator $g^{(h)}$ contains only momenta k of $O(2^h p_F)$ for $h \le 0$ while the propagator $g^{(1)}$ contains all (and only) large momenta (*i.e.* the *ultraviolet* part of the propagator $g(x)$). The representation (2.7) is called a *quasi particles* representation of the propagator and the quantities $\underline{\omega}\, p_F$ are called a *quasi particles* momenta. The function $g^{(h)}_{\underline{\omega}}$ is the "*quasi–particle propagator on scale h*".

After extracting the exponentials $e^{i\,\underline{\omega}\,p_F\,\underline{x}}$ from the propagators the Fourier transforms $\hat g^{(h)}_{\underline{\omega}}(k)$ of $g^{(h)}_{\underline{\omega}}(\underline{x})$ will no longer be oscillating on the scale of p_F and the variable k will have the interpretation of "momentum measured from the Fermi surface".

The mentioned divergences are *still present* because we do not yet relate λ and ν: they will be eliminated temporarily by introducing an *infrared cut–off*: *i.e.* by truncating the sum in (2.7) to $h \ge -R$. We then proceed keeping in mind that *we must get results which are uniform as* $R \to \infty$: this will be eventually possible only if ν is suitably fixed as a function of λ.

Writing $g(x) = Z_1^{-1} g^{(1)}(x) + Z_1^{-1} g^{(\le 0)}(x)$ with $Z_1 \stackrel{def}{=} 1$ and $g^{(\le m)}$ being defined in general, see (2.7), as:

2.9
$$g^{(\le m)}(x) = \sum_{h=-R}^{m} \sum_{\underline{\omega}=\pm 1} 2^h e^{i\,\underline{\omega}\,p_F\,\underline{x}}\, g^{(h)}_{\underline{\omega}}(2^h p_F x), \qquad m \le 0 \tag{2.9}$$

each graph can now be decomposed as a sum of graphs each of which with internal lines carrying extra labels "1" and "$\underline{\omega}$" or "≤ 0" and "$\underline{\omega}$" (signifying that the value of the graph has to be computed by using the propagator ${Z_1}^{-1} g^{(1)}_{\underline{\omega}}(x-x')$ or ${Z_1}^{-1} g^{(\le 0)}_{\underline{\omega}}(x-x')$ for the line in question, if it goes from x' to x).

We now define *clusters of scale* 1: a "cluster" on scale 1 will be any set C of vertices connected by lines bearing the scale label 1 and which are maximal in size (*i.e.* they are not part of larger clusters of the same type). *Wavy lines are regarded as bearing a scale label* 1. The graph is thus decomposed into smaller graphs formed by the clusters and connected by lines of scale ≤ 0: it is convenient to visualize the clusters as enclosed into contours that include the vertices of each cluster as well as all the lines that connect two vertices of the same cluster. The latter can be naturally called *lines internal to the cluster* C.

The integrated value of a graph will be represented, up to a sign which can be determined as described above, as a sum over the quasi particles labels $\underline{\omega}$ of the cluster lines and

as an integral over the locations of the inner vertices of the various clusters lines. The integrand is a product between
(a) the *kernels* K_{C_i} associated with the clusters C_i and depending *only* on the locations of the vertices inside the cluster C_i which are extremes of lines external to the cluster and on the quasi particles labels $\underline{\omega}$ of the lines that emerge from it,[1] and
(b) the propagators $Z_1^{-1} g^{(\le 0)}_{\underline{\omega}}$ corresponding to the lines that are external to the clusters (in the sense that they have at least one vertex not indide the cluster).

We now look at the clusters C that have just $|C| = 2$ or $|C| = 4$ external lines and that are therefore associated with kernels $K_C(\{x_j, \underline{\omega}_j\}_{j=1,2}; C)$ or $K_C(\{x_i, \underline{\omega}_i\}_{i=1,\dots,4}; C)$. Such kernels, by the structure of the propagators, see (2.7) and (2.9), will have the form:

$$K_C = e^{ip_F(\sum_j \underline{\omega}_j x_j)} \overline{K}_C(\{x_j, \underline{\omega}_j\}_{j=1,2,\dots,|C|}) \tag{2.10}$$

where x_j are vertices of the cluster C to which the entering and exiting lines are attached; the cluster may contain more vertices than just the ones to which the external lines are attached: the positions of such "extra" vertices must be considered as integration variables (and as integrated), and a sum is understood to act over all the quasi particles labels of the internal lines (consistent with the values of the external lines $\underline{\omega}_j$'s).
If $|C| = 2, 4$ we write the *Fourier transform* at $k = (k_0, \underline{k})$ of the kernels $\overline{K}_C(\dots)$:

$$\begin{aligned} &Z_1 2^{-1} \nu_C^{(1)} \delta_{\underline{\omega}_1, \underline{\omega}_2} + Z_1(-ik_0 \zeta_C^{(1)} + \underline{\omega}\, \underline{k}\, \alpha_C^{(1)})\, \delta_{\underline{\omega}_1, \underline{\omega}_2} + \text{"remainder"} \\ &Z_1^2 \lambda_C^{(1)} \delta_{\underline{\omega}_1 + \underline{\omega}_2 + \underline{\omega}_3 + \underline{\omega}_4 = 0} + \text{"remainder"} \end{aligned} \tag{2.11}$$

where the first equation ($|C| = 2$) is a function of one k only while the second equation ($|C| = 4$) depends on four momenta k: one says that the remainders are obtained by "subtracting from the kernels their values at the Fermi surface" or by collecting terms tha *do not conserve* the quasi particles momenta (like terms with $\delta_{\underline{\omega}_2, -\underline{\omega}_2}$ in the first equation or with $\underline{\omega}_1 + \dots \underline{\omega}_4 \neq 0$ in the second).
The remainder contains various terms which do not have the form of the terms explicitly written in (2.11): a form which could be as simple as $\underline{\omega} \cdot \underline{k}\, \delta_{\underline{\omega}_1, -\underline{\omega}_2}$ but that will in general be far more involved.
In evaluating graphs we imagine, as described, them as made with clusters and that the graph value is obtained by integrating the product of the values of the kernels associated with the graph times the product of the propagators of the lines that connect different clusters.
Furthermore we imagine to attach to each cluster with 2 external lines a label indicating that it contributes to the graph value with the first term in the decomposition in (2.11) only (which is the term proportional to $\nu^{(1)}$), or with the second term (which is proportional to $(-ik_0\zeta^{(1)} + \underline{\omega}\, \underline{k}\, \alpha^{(1)})$ or with the remainder. This is easily taken into account by attaching to the cluster an extra label $1, 2$ or r.
Likewise we imagine to attach to each cluster with 4 external lines a label indicating that it contributes to the value with the first term in the decomposition in (2.11) only (which is the term proportional to $\lambda^{(1)}$), or with the remainder. This is again easily taken into account by attaching to the cluster an extra label $1, r$.

The label r stands for "remainder term" or "irrelevant term", however irrelevant does not mean neglegible, as usual in the renormalization group nomenclature, (on the contrary they are in a way the most important terms).
The next idea is to collect together all graphs with the same clusters structure, *i.e.* which become identical once the clusters with 2 or 4 external lines are "shrunk" to points. Since

[1] By definition the kernel K_C also involves integration over the locations of its inner vertices and the sum over the quasi particle momenta of the inner propagators.

the internal structure of such graphs is different this means that we are collecting together graphs of different perturbative order.

In this way we obtain a representation of the Schwinger functions that is *no longer a power series* representation and the evaluation rules for graphs in which *single vertex subgraphs* (or *single node subgraphs*) with 2 or 4 external lines have a new meaning. Namely a four external lines vertex will mean a quantity $Z_1^2\lambda'$ equal to the sum of $Z_1^2\lambda_C^{(1)}$ of all the values of the clusters C with 4 external lines and with label 1.

The 2 external lines nodes will mean

2.12
$$e^{i(\underline{\omega}_1\underline{x}_1-\underline{\omega}_2\underline{x}_2)p_F}\delta_{\underline{\omega}_1,\underline{\omega}_2}Z_1(\nu'+\zeta'\partial_t-i\alpha'\,\underline{\omega}\,\partial_{\underline{x}})\delta(x_1-x_2)) \tag{2.12}$$

where again ν' or ζ', α' are the sum of the contributions from all the graphs with 2 external lines and with label 0 or 1 respectively.

It is convenient to define $\delta' = \zeta' - \alpha'$ and to rewrite the 2 external lines nodes contributions to the product generating the value of a graph simply as:

2.13
$$e^{i(\underline{\omega}_1\underline{x}_1-\underline{\omega}_2\underline{x}_2)p_F}\,\delta_{\underline{\omega}_1,\underline{\omega}_2}\,Z_1\,(2\,\nu'+\delta'\,\partial_t+\alpha'\,(\partial_t-\underline{\omega}\cdot\partial_{\underline{x}}))\,\delta(x_1-x_2) \tag{2.13}$$

One then notes that this can be represented graphically by saying that 2 external lines nodes in graphs which do not carry the label r can contribute in 3 different ways to the product determining the graph value. The 3 ways can be distinguished by a label 0, $1'$ and z corresponding to the three addends in (2.13).

Any graph without z–type of nodes can be turned into a graph which contains an arbitrary number of them, on each line connecting the clusters. And this amounts to saying that we can compute the series by imposing that there is *not even a single vertex with two external lines and with label z* simply by modifying the propagators of the lines connecting the graphs: changing them from $Z_1^{-1}g^{(\le 0)}$ to $Z_0^{-1}g^{(\le 0)}$ with

2.14
$$Z_0 = Z_1(1+\alpha') \tag{2.14}$$

This can be seen either elementarily by remarking that adding values of graphs which contains chains of nodes with label z amounts to summing a geometric series (*i.e.* precisely the series $\sum_{k=0}^{\infty}(-1)^k(\alpha')^k = (1+\alpha')^{-1}$ or, much more easily, by recalling that the graphs are generated by a formal functional integral over Grassmanian variables and checking (2.14) from this remark without any real calculation, see [BG95]. In the first approach care is needed to get the correct relation (2.14) and it is wise to check it first in a few simple cases (starting with the "linear" graphs which only contain nodes with one entering line and one exiting line, see Fig. 1: the risk is to get $Z_0 = Z_1\,(1-\alpha')$ instead of (2.14)).[2]

Correspondingly we set:

[2] It is at this point that using the sharply discontinuous χ–functions would cause a problem. In fact if one uses the smooth decomposition (2.14) is *no longer correct*: namely it would become $Z_0 = Z_1\,(1+\chi^{(0)}(\underline{k})\,\alpha')$ with the consequence that Z_1 would no longer be a constant. At this point there are two possible ways out: the first is to live with a Z_0 which dpends on k and with the fact that the quantities introduced below Z_j, $j \le -1$, will also be k–dependent; this is possible but it is perhaps too different from what one is used to in the phenomenological renormalization group approaches in which quantities like Z_j are usually constants. The other possibility is to modify the propagator on scale 0 from $Z_0^{-1}g^{(\le 0)}(\underline{k})$ to $g^{(\le 0)}(\underline{k})\,(1+\alpha')/(1+\chi^{(0)}(\underline{k})\,\alpha')$. The second choice implies that $g^{(\le h)}$ will no longer be exactly $\chi^{(h)}(\underline{k})/(-ik_0+\underline{\omega}\cdot\underline{k})$ but it will be gradually modified as h decreases and the modification has to be computed step by step. This is *also* unusual in the phenomenological renormalization group approaches: the reason being simply that in such approaches the decomposition with sharp discontinuities is always used. The latter is not really convenient if one wants to make estimates of large order graphs. Here this will not be a poblem for us because we shall not do the technical work of deriving estimates. In [BG90] as well as in [BGPS] the second choice has been adopted.

2.15
$$\nu^{(0)} = 2\frac{Z_0}{Z_1}\nu', \qquad \delta^{(0)} = \frac{Z_0}{Z_1}\delta', \qquad \lambda^{(0)} = \frac{Z_0^2}{Z_1^2}\lambda' \tag{2.15}$$

We can now *iterate the analysis*: we imagine writing the propagators of the lines connecting the clusters so far considered and that we shall call *clusters of scale* 1 as:

2.16
$$\frac{1}{Z_0}g^{(\le 0)} = \frac{1}{Z_0}g^{(0)} + \frac{1}{Z_0}g^{(\le -1)} \tag{2.16}$$

and proceed to decompose all the propagators of lines outside the clusters of scale 1 into propagators of scale 0 or of scale ≤ -1.

In this way, imagining all clusters of scale 0 as points, we build a new level of clusters (whose vertices are either vertices or clusters of scale 0): they consist of maximal sets of clusters of scale 1 *connected* via paths of lines of scale 0.

Proceeding in the same way as in the above "step 1" we represent the Schwinger functions as sums of graph values of graphs built with clusters of scale 0 connected by lines with propagators on scale ≤ -1 given by $Z_0^{-1}g^{(\le -1)}$ and with the clusters carrying labels $1, 2$ or r. Again we rearrange the 2–external lines clusters with labels $1, 2$ as in (2.13) introducing the parameters $\delta', \lambda', \alpha', \nu'$ and graphs with nodes of type z by defining α' in an analogous way as the previous quantity with the same name (relative to the scale 1 analysis).

The one vertex nodes of such graphs with 2 or 4 external lines of scale ≤ -1 will contribute to the product defining the graph value, a factor $Z_{-1}2\nu^{(-1)}$ or $Z_{-1}\delta^{(-1)}$ or $Z_{-1}^2\lambda^{(-1)}$ (while the propagators in the clusters of scale 1 and the 2 or 4 nodes with two lines of scale 0 emerging from them retain the previous meaning). Again one sets:

2.17
$$Z_{-1} = Z_0(1 + \alpha') \tag{2.17}$$

and correspondingly we set:

2.18
$$\nu^{(-1)} = 2\frac{Z_{-1}}{Z_0}\nu', \qquad \delta^{(-1)} = \frac{Z_{-1}}{Z_0}\delta', \qquad \lambda^{(-1)} = \frac{Z_{-1}^2}{Z_0^2}\lambda' \tag{2.18}$$

and now we shall only have graphs with 2 or 4 external lines clusters which carry a label $0, 1'$ or r as in the previous analysis of the scale 1 and the propagators connecting clusters of scale 0 changed from $Z_0^{-1}g^{(\le -1)}$ to $Z_{-1}^{-1}g^{(\le -1)}$.

Having completed the step 0 we then "proceed in the same way" and perform "step -1" and so on.

One can wander why the choice of the scaling factor 2 in (2.13) and (2.18) multiplying the ratio of the renormalization factors in the definition of the new ν_j or, for that matter, why the choice of 1 for the definition of the new δ_j, λ_j: these are dimensional factors that come out naturally and any attempt at modifying the above choices leads to a beta function, defined below, which is not uniformly bounded as we remove the infrared cut–off. In other words: different scalings can be considered but there is only one which is useful. It could also be found by using arbitrary scaling and then look for which one the estimates needed to get a convergent expansion can be made.

The conclusion is a complete rearrangement of the perturbation expansion which is now expressed in terms of graphs which bear various labels and, most important, contain propagators that bear a scale index which gives us information on the scale on which they are sizably different from 0. The procedure, apart from convergence problems, leads us to define recursively a sequence $\lambda^{(j)}, \delta^{(j)}, \nu^{(j)}, Z_j$ of constants each of which is a sum of a formal power series involving values of graphs with 2 or 4 external lines. The quantities $\underline{g}_j = (\lambda^{(j)}, \delta^{(j)}, \nu^{(j)})$ can be called the *running coupling constants* while Z_j can be called the *running wave function renormalization constants*: here $j = 1, 0, -1, -2, \ldots$.

Of course all the above is nothing but algebra, made simple by the graphical representation of the objects that we wish to compute. The reason why it is of any interest is that, since the construction is recursive, one derives expressions of the $\underline{g}_j, Z_j$ in terms of the $\underline{g}_n, Z_n$ with $n > j$:

2.19
$$\begin{aligned} \frac{Z_{j+1}}{Z_j} &= 1 + B'_j(\underline{g}_{j+1}, \underline{g}_j, \dots, \underline{g}_0) \\ \underline{g}_j &= \Lambda_j\, \underline{g}_{j+1} + \underline{C}_j(\underline{g}_{j+1}, \underline{g}_j, \dots, \underline{g}_0) \end{aligned} \tag{2.19}$$

where Λ_j is a matrix:

2.20
$$\Lambda_j = \begin{pmatrix} (Z_h/Z_{h-1})^2 & 0 & 0 \\ 0 & (Z_h/Z_{h-1}) & 0 \\ 0 & 0 & 2\,(Z_h/Z_{h-1}) \end{pmatrix} \tag{2.20}$$

and the functions $B'_j, \underline{C}_j$ are given by power series, so far formal, in the running couplings. The expression of Z_{j+1}/Z_j can be used to eliminate such ratios in the second relation of (2.19) which therefore becomes

2.21
$$\begin{aligned} \frac{Z_{j+1}}{Z_j} &= 1 + B'_j(\underline{g}_{j+1}, \underline{g}_j, \dots, \underline{g}_0) \\ \underline{g}_j &= \Lambda\, \underline{g}_{j+1} + \underline{B}_j(\underline{g}_{j+1}, \underline{g}_j, \dots, \underline{g}_0) \end{aligned} \tag{2.21}$$

where Λ is the diagonal matrix with diagonal $(1, 1, 2)$. The scalar functions B'_j and the three components vector functions $\underline{B}_j = (B_{j,1}, B_{j,2}, B_{j,3})$ are called the *beta functional* of the problem.

There are two key points, which are nontrivial at least if compared to the above simple algebra and which we state as propositions

Proposition 1 *(regularity and boundedness of the beta function): Suppose that there is $\varepsilon > 0$ such that $|\underline{g}_j| < \varepsilon, |Z_j/Z_{j-1} - 1| < \varepsilon$ for all $j \le 1$ then if ε is small enough the power series defining the beta functionals converge. Furthermore the functions B_j, B'_j are uniformly bounded and have a dependence on the arguments with label $j+n$ exponentially decaying as n grows, namely there exist constants D, κ such that if $\underline{G} = (\underline{g}_{j+1}, \dots, \underline{g}_0)$ and $\underline{G}' = (\underline{g}'_{j+1}, \dots, \underline{g}'_0)$ with $\underline{G}$ and $\underline{F}$ differing only by the $(j+n)$–th "component" $\underline{d} = \underline{g}'_{j+n} - \underline{g}_{j+n} \neq \underline{0}$, then for all $j \le 0$ and all $n \ge 0$*

2.22
$$\begin{aligned} &|B_j(\underline{G})|,\ |B'_j(\underline{G})| \le D\varepsilon^2 \\ &|B_j(\underline{G}') - \underline{B}_j(\underline{G})|,\ |B'_j(\underline{G}') - \underline{B}'_j(\underline{G})| \le D\, e^{-\kappa n}\, \varepsilon\, |\underline{d}| \end{aligned} \tag{2.22}$$

if ε is small enough: i.e. the "memory" of the "beta functionals" B_j, B'_j is short ranged. The Schwinger functions are expressed as convergent power series in $\underline{g}_j$ in the same domain $|\underline{g}_j| < \varepsilon$.

The difficult part of the proof of the above proposition is to get the convergence of the series under the hypotheses $|\underline{g}_j| < \varepsilon$, $|Z_j/Z_{j-1} - 1| < \varepsilon$ for all j: this is possible because the system is a fermionic system and one can collect the contributions of all graphs of a given order k into a few, *i.e.* not more than an exponential in k, groups each of which gives a contribution that *is expressed as a determinant which can be estimated without really expanding it* into products of matrix elements (which would lead to bounding the order k by a quantity growing with $k!$) by making use of the Gram–Hadamard inequality. Thus the $k!^{-1}$ that is in the definition of the values compensates the number of labels that one can put on the trees and the number of Feynman graphs that is also of order $k!$ is controlled by their representability as determinants that can be bounded without

generating a $k!$ via the Hadamard inequality. The basic technique for achieving these bounds is well established after the work [Le87]. A second non trivial result is

Proposition 2 *(short range and asymptotics of the beta function):*
Let $\underline{G}^0 = (\underline{g}, \underline{g}, \dots, \underline{g})$ *with* $\underline{g} = (\lambda, \delta, \nu)$ *then the function* $\underline{B}_j(\underline{G}^0)$ *defines an analytic function of* $\underline{g}$*, that we shall call "beta functional", by setting*

$$\underline{\beta}(\underline{g}) = \lim_{j \to -\infty} \underline{B}_j(\underline{G}^0) \tag{2.23}$$

for $|\underline{g}| < \varepsilon$*. The limit is reached exponentially* $|\underline{\beta}(\underline{g}) - \underline{B}_j(\underline{G}^0)| < \varepsilon^2 D e^{-\kappa |j|}$*, for some* $\kappa > 0, D > 0$ *provided* $|\underline{g}| < \varepsilon$*.*

Finally the key result, [BG90], [BGPS], is

Proposition 3 *(vanishing of the beta function): If* $\underline{g} = (\lambda, \delta, 0)$ *then the functions* $\underline{\beta}(\underline{g}) = \underline{0}$ *provided* $|\underline{g}| < \varepsilon$*. Furthermore for some* $D, \kappa > 0$ *it is, for all* $j \leq 0$

$$\begin{aligned} &B_{j3}(\underline{g}_{j+1}, \dots, \underline{g}_0) = \nu_{j+1} \lambda_{j+1}^2 B'_{j3}(\underline{g}_{j+1}, \dots, \underline{g}_0) + e^{\kappa j} B''_{j3}(\underline{g}_{j+1}, \dots, \underline{g}_0) \\ &|B'_{j3}(\underline{g}_{j+1}, \dots, \underline{g}_0)| < D, \qquad |B''_{j3}(\underline{g}_{j+1}, \dots, \underline{g}_0)| < D\, \varepsilon^2 \end{aligned} \tag{2.24}$$

provided, for $h = 0, \dots, j+1$, $|\underline{g}_h| < \varepsilon$*.*

The above propositions are proved in [BG90], [BGPS], [BM00]. The vanishing of $\underline{\beta}(\lambda, \delta, 0)$ is proved in a rather indirect way. We proved that the function $\underline{\beta}$ is the *same* for the model (2.1) and for a similar model, the *Luttinger model*, which is exactly soluble; but which can be also studied with the technique described above: and the only way the exactly soluble model results could hold is to have $\underline{\beta} = \underline{0}$.

The vanishing of the beta function seems to be a kind of Ward identity: it is easy to prove it directly if one is willing to accept a formal proof. This was pointed out, after the work [BG90], in other papers and it was believed to be true probably much earlier in some equivalent form, see [So79]; note that the notion of the beta function is *intrinsic to the formalism* of the renormalization group and therefore a precise conjecture on it could not even be stated before the '70s; but of course the existence and importance of infinitely many identities had already been noted.

Given the above propositions one shows that "things go as if" the recursion relation for the running couplings was, up to exponentially small corrections, a simple memoryless evolution $\underline{g}_{j-1} = \underline{\beta}(\underline{g}_j) + O(e^{-\kappa |j|})$: the propositions say in a precise way that this is asymptotically, as $j \to -\infty$, true. This tells us that the running couplings λ_j, δ_j stay constant (because β_1, β_2 vanish): however they in fact tend to a limit as $j \to -\infty$ exponentially fast because of the corrections in the above propositions, provided we can guarantee that also $\nu_j \xrightarrow[j \to -\infty]{} 0$ exponentially fast and that the limits of $\lambda_,, \delta_j$ do not exceed ε (so that the beta functionals and the beta function still make sense).

It is now important to recall that we can adjust the initial value of the chemical potential.[3] This freedom corresponds to the possibility of changing the chemical potential 'correction" ν in (2.1) and tuning its value so that $\nu_h \to 0$ as $h \to -\infty$.

Informally if ν_0 is chosen "too positive" then ν_j will grow (exponentially) in the positive direction (becoming larger than ε, a value beyond which the series that we are using become meaningless); if ν_0 is chosen "too negative" the ν_j also will grow (exponentially)

[3] Which is a "*relevant operator*", in the sense that if regarded as a running coupling it is roughly multiplied by 2 at each change of scale, *i.e.* $\nu_{j-1} \sim 2\nu_j$.

in the negative direction: so there is a unique choice such that ν_j can stay small (and, *actually*, it can be shown to converge to 0.[4]

The vanishing of the beta function gives us the existence of a sequence of running couplings $\underline{g}_j = (\lambda_j, \delta_j, \nu_j)$ which converge exponentially fast to $(\lambda_{-\infty}, \delta_{-\infty}, 0)$ as $j \to -\infty$ if ν_0 are conveniently chosen: and one can prove that $\lambda_{-\infty}, \delta_{-\infty}, \nu_0$ are analytic in λ for λ small enough, [BGPS].

In this way one gets a convergent expansion of the Schwinger functions: which leads to an essentially complete theory of the one dimensional Fermi gas with spin zero and short range interaction.

§3. The conceptual scheme of the renormalization group approach followed above.

The above schematic exposition of the method is a typical example of how one tries to apply the multiscale analysis that is commonly called a "renormalization group approach":

(1) one has series that are easily shown to be finite order by order possibly provided that some free parameters are suitably chosen ("formal renormalizability theory": this is the proof in [LW60] that if ν_h in (2.2) are suitably chosen we obtain a well defined perturbation series in powers of λ.

(2) However the series even when finite term by term come with poor bounds which grow at order k as $k!$ which, nevertheless are often non trivial to obtain (although this not so in the case (2.1) discussed here unlike the case discussed in the next sections).

(3) One then tries to reorganize the series by leaving the original parameters (λ, ν) in the present case as μ is fixed) as independent parameters and collecting terms together. The aim being to show that they become very *convergent power series in a sequence of new parameters*, the "running couplings" $\nu^{(h)}, \delta^{(h)}$ and $\lambda^{(h)}$ in the present case, *under the assumption that such parameters* are small (they are functions, possibly singular, of the initial parameters of the theory, λ, ν in the case (2.1), as δ has to be imagined to be 0).

(4) The running couplings, essentially by construction, also verify a recursion relation that makes sense *again* under the assumption that the parameters are small. This relation allows us to express (if it makes sense) successively the running couplings in terms of the preceding ones: the running couplings are ordered into a sequence by "scale labels" $h = 1, 0, -2, \ldots$. The recursion relation is interpreted as an evolution equation for a dynamical system (a map defined by the beta function(al)): it generates a "renormalization group trajectory" (the sequence $(\lambda_h, \delta_h, \nu_h)$ out of the original parameters λ, ν present in (2.1), as δ has to be taken as 0).

(5) One then shows that *if the free parameters in the problem*, (*i.e.* λ, ν in (2.1)) *are conveniently chosen*, then the recursion relation implies that the trajectory stays bounded and small, thus giving a precise meaning to (2.2)), and actually the limit relation holds $(\lambda^{(h)}, \delta^{(h)}, \nu^{(h)}) \xrightarrow[h \to -\infty]{} (\lambda_\infty, \delta_\infty, 0)$ (this is achieved in the above Fermionic problem by fixing ν as a suitable function of λ, see [BGPS].

(6) Hence the whole scheme is self–consistent and it remains to check that the expressions that one thus attributes to the sum of the series are indeed solutions of the problem that has generated them: not unexpectedly this is the easy part of the work, because we have always worked with formal solutions which "only missed, perhaps, to be convergent".

(7) The first step, *i.e.* going to scale 0 is different from the others as the propagators have no ultraviolet cut off (see the graph of $\chi^{(1)}$ in Fig. 2). Although there are no

[4] A simplified analysis is obtained by "neglecting memory corrections" *i.e.* using as a recursion relation $\underline{g}_j = \Lambda \underline{g}_{j+1} + \underline{\beta}(\underline{g}_{j+1})$ with $\underline{\beta}(\underline{g})$ verifying (2.24): this gives that $\lambda_j, \delta_j \xrightarrow[j \to -\infty]{} (\lambda_{-\infty}, \delta_{-\infty})$ exponentially fast and $\nu_j \xrightarrow[j \to -\infty]{} 0$ exponentially fast provided ν_0 is suitably chosen in terms of λ_0, δ_0: otherwise everything diverges.

ultraviolet divergences the control of this first step offers surprising difficulties (due to the fact that in the direction of k_0 the decay of the propagators is slow making various integrals improperly convergent): the analysis is done in [BGPS] and [GS93].

Note that the above scheme leaves room for the possibility that the running couplings rather than being analytic functions of a few of the initial free parameters are singular: this does not happen in the above fermionic problem because some components of the beta function vanish identically: this is however a peculiarity of the fermionic models. In other applications to field theory, and particularly in the very first example of the method which is the hierarchical model of Wilson, this is by far not the case and the perturbation series are *not analytic* in the running couplings but *just asymptotic* in the actual free parameters of the theory. The method however "reduces" the perturbation analysis to a recursion relation in small dimension (namely 3 in the case (2.1)) which is also usually easy to treat heuristically. The $d = 2$ ground state fermionic problem (*i.e.* (2.1) in 2 space dimensions) provides, however, an example in which even the heuristic analysis is not easy.

In the following section we discuss another problem where the beta function does not vanish, but one can guarantee the existence of a bounded and small solution for the running couplings thanks to a "gauge symmetry" of the problem. This is an interesting case as the theory has *no free parameters* so that it would not be possible to play on them to find a bounded trajectory for the renormalization group running constants. This also illustrates another very important mechanism that can save the method in case there seemed to be no hope for its use, namely a symmetry that magically eliminates all terms that one would fear to produce "divergences" in formal expansions. Again the case studied is far from the complexity of gauge field theory because it again leads to the result that the perturbation series itself is summable (unlike gauge field theories which can only yield asymptotic convergence): but it has the advantage of being a recognized difficult problem and therefore is a nice illustration of the role of symmetries in the resummation of (possibly) divergent series and the power of the renormalization group approach in dealing with complex problems.

§4. The KAM problem.

Consider d rotators with angular momentum $\underline{A} = (A_1, \dots, A_d) \in R^d$ and positions $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in T^d = [0, 2\pi]^d$; let $J > 0$ be their inertia moment and suppose that $\varepsilon f(\underline{\alpha})$ is the potential energy in the configuration $\underline{\alpha}$, which we suppose to be an even trigonometric polynomial (for simplicity) of degree N. Then the system is Hamiltonian with Hamiltonian function

4.1
$$\mathcal{H} = \frac{1}{2J}\,\underline{A}^{\,2} + \varepsilon f(\underline{\alpha}) \tag{4.1}$$

giving rise to a model called "Thirring model".[5]

[5] (1) The global canonical transformations $\mathcal{C}$ of $R^d \times T^d$ with generating functions $S(\underline{A}, \underline{\alpha}) = N\underline{A}' \cdot \underline{\alpha} + \underline{\gamma}(\underline{\alpha}) \cdot \underline{A}' + \varphi(\underline{\alpha})$ parameterized by an integer components non singular matrix N, and analytic functions $\underline{g}(\underline{\alpha}), f(\underline{\alpha})$ leave invariant the class of Hamiltonians of the form $H = (\underline{A}, M(\underline{\alpha})\underline{A})/2 + \underline{A} \cdot \underline{g}(\underline{\alpha}) + f(\underline{\alpha})$. The subgroup $CL_d(R)$ of the global canonical coordinate trransformations $\mathcal{C}$ was (remarked and) used by Thirring so that (4.1) is called the "Thirring model", see [Th83].
(2) The function $\underline{H}_\varepsilon(\underline{\psi})$ in (4.2) must have zero average over $\underline{\psi}$ or, if $\underline{\psi} \to \underline{\psi} + \underline{\omega}_0 t$, over time: hence the surviving quasi periodic motions can be parameterized by their spectrum $\underline{\omega}_0$ or, equivalently, by their average action $\underline{A}_0 = J\underline{\omega}_0$.. The "spectral dispersion relation" between the average action $\underline{A}_0$ and the frequency spectrum is *not twisted* by the perturbation. Furthermore the function $\varepsilon_(\underline{\omega}_0, J)$ can be taken monotonically increasing J: J^{-1} is called the "*twist rate*. The latter two properties motivated the name of "*twistless motions*" given to the quasi periodic motions of the form (4.2) for Hamiltonians like (4.1).
(3) The invariance under the group $CL_d(R)$ has been used widely in the numerical studies of the best treshold value $\varepsilon(\underline{\omega}_0, J)$ and a deeper analysis of this group would be desirable, particularly a theory of

For $\varepsilon = 0$ motions are quasi periodic (being $t \to (\underline{A}_0, \underline{\alpha}_0 + \underline{\omega}_0 t)$ with $\underline{\omega}_0 = J^{-1}\underline{A}_0$) and their "spectrum" $\underline{\omega}_0$ fills the set $S_0 \equiv R^d$ of *all* vectors $\underline{\omega}_0$: there is a 1-to-1 correspondence between the spectra $\underline{\omega}_0$ and the angular momenta $\underline{A}_0$.

Question: If $\varepsilon \neq 0$ can we find, given $\underline{\omega}_0 \in S_0$ a perturbed motion, i.e. a solution of the Hamilton equations of (4.1), which has spectrum $\underline{\omega}_0$ and, as $\varepsilon \to 0$, reduces with continuity to the unperturbed motion with the same spectrum? or less formally: which among the possible spectra $\underline{\omega} \in S_0$ survives perturbation?

Analytically this means asking whether two functions $\underline{H}_\varepsilon(\underline{\psi})$, $\underline{h}_\varepsilon(\underline{\psi})$ on T^d exist, are divisible by ε and are such that if $\underline{A}_0 = J\,\underline{\omega}_0$ and if we set

$$\begin{aligned} \underline{A} &= \underline{A}_0 + \underline{H}_\varepsilon(\underline{\psi}) \\ \underline{\alpha} &= \underline{\psi} + \underline{h}_\varepsilon(\underline{\psi}) \end{aligned}, \qquad \underline{\psi} \in T^d \tag{4.2}$$

then $\underline{\psi} \to \underline{\psi} + \underline{\omega}_0 t$ yields a solution of the equations of motion for ε small enough.

It is well known that *in general* only "non resonant" spectra can survive: for instance (*KAM theorem*) those which verify, for some $\tau, C > 0$

$$|\,\underline{\omega}_0 \cdot \underline{\nu}\,|^{-1} < C|\,\underline{\nu}\,|^\tau \qquad \forall\, \underline{\nu} \in Z^d = \{\textit{integer vectors}\},\ \underline{\nu} \neq \underline{0} \tag{4.3}$$

for some $C, \tau > 0$ (*Diophantine vectors*) and we restrict for simplicity to such vectors. Furthermore given $\underline{\omega}_0$ the perturbation size ε has to be small enough: $|\varepsilon| \le \varepsilon_0(\underline{\omega}_0, J)$.

To find $\underline{H}_\varepsilon$, $\underline{h}_\varepsilon$ we should solve the equation (setting $J \equiv 1$),

$$(\,\underline{\omega}_0 \cdot \underline{\partial}_{\underline{\psi}}\,)^2\, \underline{h}\,(\underline{\psi}\,) = -\varepsilon\,(\,\underline{\partial}_{\underline{\alpha}} f)(\underline{\psi} + \underline{h}(\underline{\psi}\,)) \tag{4.4}$$

and if such $\underline{h}$ is given then, setting $\underline{h}_\varepsilon(\underline{\psi}) = \underline{h}(\underline{\psi}\,)$, $\underline{H}_\varepsilon(\underline{\psi}\,) = (\,\underline{\omega}_0 \cdot \underline{\partial}\,)\,\underline{h}(\underline{\psi}\,)$ one checks that (4.2) has the wanted property (*i.e.* $\underline{\psi} \to \underline{\psi} + \underline{\omega}_0 t$ is a motion for (4.1)).

To an exercized eye (4.4) defines the expectation value of the 1–particle Schwinger function of the euclidean field theory on the torus T^d for two vector fields $\underline{F}^{\pm}(\underline{\psi}\,) = (F_1^{\pm}(\underline{\psi}\,), \dots, F_d^{\pm}(\underline{\psi}\,))$ whose free propagator is

$$\langle F_\ell^\sigma(\,\underline{\psi}\,)\, F_m^{\sigma'}(\,\underline{\psi}\,')\rangle = \delta_{\sigma,-\sigma'}\,\delta_{\ell,m}\frac{-1}{(2\pi)^d}\sum_{\underline{\nu}\neq\underline{0}}\frac{e^{i(\underline{\psi}-\underline{\psi}\,')\cdot\underline{\nu}}}{(\,\underline{\omega}_0\cdot\underline{\nu}\,)^2} \tag{4.5}$$

and the interaction Lagrangian is, see [Ga95]

$$\mathcal{L}(F) = \varepsilon\int_{T^d}\underline{F}^{+}(\,\underline{\psi}\,)\cdot\underline{\partial}_{\underline{\alpha}}(\,\underline{\psi} + \underline{F}^{-}(\,\underline{\psi}\,))\,d\,\underline{\psi} \tag{4.6}$$

If $P_0(d\,\underline{F}\,)$ the "functional integral" defined by Wick's rule with propagator (4.5) then

$$\underline{h}\,(\,\underline{\psi}\,) = \frac{\int \underline{F}^{-}(\,\underline{\psi}\,)\, e^{\varepsilon\int_{T^d}\underline{F}^{+}(\underline{\psi})\cdot\underline{\partial}_{\underline{\alpha}} f(\underline{\psi}+\underline{F}^{-}(\underline{\psi}))}\, P_0(d\,\underline{F}\,)}{\int e^{\varepsilon\int_{T^d}\underline{F}^{+}(\underline{\psi})\cdot\underline{\partial}_{\underline{\alpha}} f(\underline{\psi}+\underline{F}^{-}(\underline{\psi}))}\, P_0(d\,\underline{F}\,)} \tag{4.7}$$

At first sight this is a "sick field theory". Not only the fields $\underline{F}^{\pm}(\underline{\psi}\,)$ do not come from a positive definite propagator, hence (4.7) has to be understood as generating a formal expansion in ε of $\underline{h}$ with integrals over $\underline{F}$ being *defined* by the Wick rule, but also the theory is *non polynomial* and naively non renormalizable.

The "only" simplification is that the Feynman diagrams of (4.7) are (clearly) tree–graphs, *i.e.* loopless: this greatly simplifies the theory which, however, remains non renormalizable and non trivial (being equivalent to a non trivial problem).

its unitary representations.

It is not difficult to work out the Feynman rules for the diagrams expressing the k–th order coefficient of the power series expansion in ε of (4.7).

Consider a rooted tree with k nodes: the branches are considered oriented towards the root which is supposed to be reached by a single branch and which is not regarded as a node of the tree (hence the number of nodes and the number of branches are equal).

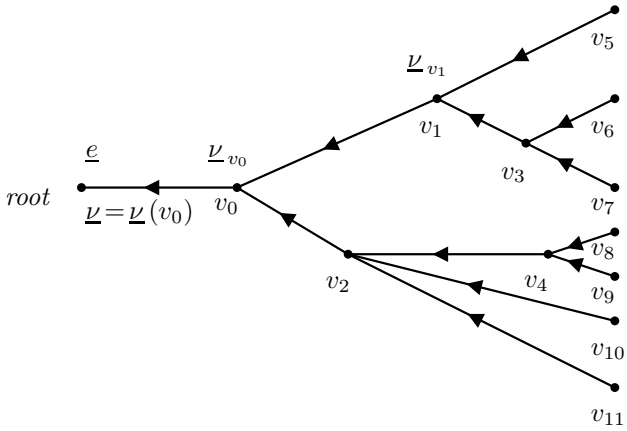

Fig. 5: A graph ϑ with $p_{v_0}=2, p_{v_1}=2, p_{v_2}=3, p_{v_3}=2, p_{v_4}=2$ and $k=12$, and some labels. The line numbers, distinguishing the lines are not shown. The lines length should be the same but it is drawn of arbitrary size. The momentum flowing on the root line is $\underline{\nu}=\underline{\nu}(v_0)=$ sum of all the nodes momenta, including $\underline{\nu}_{v_0}$.

We attach to each node (or "vertex") v of the tree a vector $\underline{\nu}_v \in Z^d$, called a "mode label", and to a line oriented from v to v' we attach a "current" or "momentum" $\underline{\nu}(v)$ and a "*propagator*" $g(\underline{\nu}(v))$:

4.8
$$\underline{\nu}(v) \stackrel{def}{=} \sum_{w\le v} \underline{\nu}_v, \qquad g(\underline{\omega}_0\cdot\underline{\nu}) \stackrel{def}{=} (\underline{\omega}_0\cdot\underline{\nu})^{-2} \tag{4.8}$$

if v is the "first node" of the tree, see v_0 in Fig. 5, then the momentum $\underline{\nu}(v)$ is called the total momentum of the tree.

One defines the *value* $\mathrm{Val}(\vartheta)$ of a tree ϑ decorated with the described labels as

4.9
$$\mathrm{Val}(\vartheta) = k!^{-1} \prod_{v\in \mathrm{nodes}} f_{\underline{\nu}_v} \frac{\underline{\nu}_v\cdot\underline{\nu}_{v'}}{(\underline{\omega}_0\cdot\underline{\nu}(v))^2} \tag{4.9}$$

where $f_{\underline{\nu}}$ are the Fourier coefficients of the perturbation $f(\underline{\alpha})=\sum_{\underline{\nu}} f_{\underline{\nu}} e^{i\underline{\nu}\cdot\underline{\psi}}$, with $|\underline{\nu}|\le N$, $f_{\underline{\nu}}=f_{-\underline{\nu}}$, and v' denotes the node following v and, if v is the first node of the tree (so that v' would be the root which (by our conventions) is not a vertex) then $\underline{\nu}_{v'}$ is some unit vector $\underline{e}$, see Fig. 3.

Given the above Feynman rules, which one immediately derives from (4.6), (4.7) the component along the vector $\underline{e}$, labeling the root, of the k–the order Fourier coefficient $\underline{h}^{(k)}_{\underline{\nu}}$ of the function $\underline{h}(\underline{\psi})\cdot\underline{e}$, which we write as

4.10
$$\underline{h}_{\underline{\nu}}\cdot\underline{e} = \varepsilon\,\underline{h}^{(1)}_{\underline{\nu}}\cdot\underline{e} + \varepsilon^2\,\underline{h}^{(2)}_{\underline{\nu}}\cdot\underline{e} + \dots, \tag{4.10}$$

is simply the sum of the values $\mathrm{Val}\vartheta$ *over all trees* ϑ which have total momentum $\underline{\nu}$, k nodes, and *no* branch carrying $\underline{0}$–momentum.

One can check directly that $\underline{h}(\underline{\psi})$ so defined is a formal solution of (4.4): the series (4.10) with the coefficients defined as above is called the *Lindstedt series* of the KAM problem: it was introduced, at least as a method for computing the low order coefficients $\underline{h}^{(k)}$, by Lindstedt and Newcomb in celestial mechanics problems and it was shown to be possible to all orders by Poincaré (one has to show that the algorithm generating the

series does not produce graphs with branches carrying $\underline{0}$ momentum which, by (4.9), would yield meaningless expressions for the corresponding tree values), see [Po92], vol. 3.

The number of trees of order k that do not differ only by the labeling of the lines (*i.e.* that are topologically the same) is bounded exponentially in k while the total number of trees (labels included) is, therefore, of order $k!$ times an exponential in k.

Therefore taking into account the $k!^{-1}$ in (4.8) we see that the perturbation series might have convergence problems only if there exist individual graphs whose value is too large, *e.g.* $O(k!^{\gamma})$ for some $\gamma > 0$.

Such graphs *do exist*; an example:

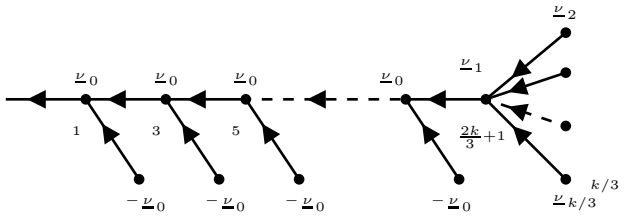

Fig. 4: A tree of order k with momentum $\underline{\nu} = \underline{\nu}_1 + \dots \underline{\nu}_{k/3}$ and value of size of order $(k/3)!^{\tau}$ if $\underline{\nu} = \underline{\nu}_1 + \dots \underline{\nu}_k$ is "as small as it can possibly be": or is "almost resonant" *i.e.* such that $\underline{\omega}_0 \cdot \underline{\nu} \sim C^{-1}|\underline{\nu}|^{-\tau}$. The tree has $k/3+1$ branches carrying momentum $\underline{\nu} = \sum_{i=1}^{k/3} \underline{\nu}_i$, *i.e.* the $k/3+$ horizontal branches. The last $k/3\ -1$ branches have momenta $\underline{\nu}_i, i = 2, \dots, k/3$ so arranged that their sum plus $\underline{\nu}_1$, *i.e.* $\underline{\nu}$, is "almost resonant".

Therefore even though the theory is loopless and its perturbation series is well defined to all orders, yet it is non trivial because the k–th order might be "too large".

This is a typical situation in "infrared divergences" due to too large propagators: in fact the "bad" graphs (like the one in Fig. 5) are such because $(\underline{\omega} \cdot \underline{\nu})^{-2}$, *i.e.* the propagator, is too large.

It is remarkable that the same strategy used in the analysis of the Fermi gas theory, *i.e.* the renormalization group approach outlined in general terms in §3, works in this case. We decompose the propagator as

4.11
$$\frac{1}{(\underline{\omega}_0 \cdot \underline{\nu})^2} = \sum_{h=-\infty}^{1} \frac{\chi^{(h)}(\underline{\omega}_0 \cdot \underline{\nu})}{(\underline{\omega}_0 \cdot \underline{\nu})^2} = \sum_{h=-\infty}^{1} 2^{-2h} g^{(h)}(2^h \underline{\omega}_0 \cdot \underline{\nu}) \tag{4.11}$$

where if $h \le 0$ we have set $\chi^{(h)}(x) \stackrel{def}{=} \chi(2^{-h}x)$ with $\chi(x) = 0$ unless x is in the interval $2^{-1} < |x| \le 1$ where $\chi(x) \equiv 1$, and $\chi^{(1)}$ is defined to be identically 1 for $|x| \ge 1$ and 0 otherwise so that $1 \equiv \sum_{h=-\infty}^{1} \chi^{(h)}(x)$: see Fig. 2 and (2.9).[6]

Given a Feynman graph, *i.e.* a tree ϑ (with the decorating labels) one replaces $g(\underline{\omega}_0 \cdot \underline{\nu})$ by the last sum in (4.11) and we obtain trees with branches bearing a "*scale label*".

[6] In this problem there is *no need* to use functions $\chi^{(h)}$ which are not as in Fig. 4, *i.e.* with smoothed out discontinuities: there is, however, a minor difficulty also in this case. The decomposition above, (4.11), can be done exactly as written only if $\underline{\omega}_0 \in R^d$ is outside a certain set of zero volume in R^d. Although already the Diophanntine property (4.3) holds only outside a set of zero volume in R^d, as is well known, the set of $\underline{\omega}_0 \in R^d$ for which what follows can be done literally as described is slightly smaller (although *still* with a complement of zero volume). However the following discussion can be repeated under the only condition that $\underline{\omega}_0$ verifies (4.3) provided one does not insist in taking a sequence of scales that are exactly equal to 2^h, $h = 1, 0, -1, \dots$ and one takes a sequence of scales that have bounded but suitably variable successive ratios. Here we ignore this problem, see [Ga94] for the cases that work as described here (first reference) and for the general case (second reference) discussed under the only "natural" condition (4.3).

We collect the graph lines into clusters of scale $\geq h$ with at least one line of scale h and define the order n of the cluster to be the number of nodes in it.

It is not difficult, see [Pö84] and [Ga94], to see that *only graphs which contain one incoming and one outgoing line with the same momentum* (hence same scale) are the source of the problem: for instance in the case of the graph in Fig. 4 all horizontal lines have the same momentum of scale $h \simeq \log k^{-\tau}$ while all the other lines have scale $O(1)$ because $|\underline{\nu}_j| \leq n$. The subgraphs that contain one incoming and one outgoing line with the same momentum of scale $h \leq 0$ have been called in [El96] *resonances*, perhaps not very appropriately given the meaning that is usually associated with the word resonance but we adopt here the nomenclature of the latter breakthrough work. The subgraphs which would be resonances but have maximal scale $h = 1$ are not considered resonances.

Therefore it is natural to collect together all graphs which contain *chains* of $1, 2, 3 \ldots$ clusters with equal incoming and outgoing momenta on scale $h \leq 0$. For instance the graph in Fig. 4 contains a chain of $k/3$ clusters, each containing a single line (namely the lines with $\underline{\nu}_0, -\underline{\nu}_0$ modes at their extremes), and one cluster with $k/3-1$ lines (the lines entering the node $2k/3+1$) and $k/3+1$ lines external to the resonant clusters, the horizontal lines. Call

4.12
$$\Gamma^{(h)}(\underline{\nu}) = \text{sum of } \varepsilon^n \text{ times the value of the clusters with } n \text{ nodes and single incoming and outgoing lines of equal momentum and scale } h \tag{4.12}$$

which can be given meaning because disregarding the incoming line we can regard the subcluster with one entering and one exiting line as a tree ϑ (or subtree) with root at the node where the exiting line ends, so that *its value* $\Gamma^{(h)}(\underline{\nu})$ *will be naturally defined* as the value of ϑ times $(\underline{\omega}_0 \cdot \underline{\nu})^{-2}$ (which takes into account the propagator of the line entering the cluster).

This leads to a rearrangement of the series for $\underline{h}$ in which the propagator of a line with momentum $\underline{\nu}$ on scale h is $\Gamma^{(h)}(\underline{\nu})$ rather than $(\underline{\omega}_0 \cdot \underline{\nu})^{-2}\chi^{(h)}(\underline{\omega}_0 \cdot \underline{\nu})$ and there are *no more* clusters with one incoming and one outgoing line of equal momentum.

Of course the graphs with k nodes give a contribution to $\underline{h}$ which is no longer proportional to ε^k because they contain quantities $\Gamma^{(h)}(\underline{\nu})$ which are (so far formally) power series in ε.

The same argument invoked above, [Pö], [Ga94], gives again that *if for some constant* R *and for all* ε *small enough one could suppose that*

4.13
$$|\Gamma^{(h)}(\underline{\nu})| \leq R\,(\underline{\omega}_0 \cdot \underline{\nu})^{-2} \tag{4.13}$$

then the series for $\underline{h}_{\underline{\nu}}$ *would be convergent for* ε *small enough.*

Furthermore if (4.13) holds for scales $0, -1, \ldots, h+1$ then one sees that there exist a_h, b_h, r_h such that

4.14
$$\Gamma^{(h)}(\underline{\nu}) = \frac{a_h}{(\underline{\omega}_0 \cdot \underline{\nu})^4} + \frac{b_h}{(\underline{\omega}_0 \cdot \underline{\nu})^3} + (1 + r_h(\underline{\nu}))\overline{g}^{(h)}(\underline{\omega}_0 \cdot \underline{\nu}) \tag{4.14}$$

with $|r_h| < R\varepsilon$, $|\overline{g}^{(h)}(\underline{\omega}_0 \cdot \underline{\nu})| < N^2(\underline{\omega}_0 \cdot \underline{\nu})^{-2}$, and we see therefore that a_h, b_h play the role of "*running couplings*": they are non trivial functions of ε but *they are the only quantities to control* because if we can show that they are bounded by εR, say, then the convergence of the perturbation series would be under control because *we are reduced essentially to the case in which no graphs with resonances are present.*

Naturally, by the principle of conservation of difficulties, the quatities a_h and b_h are given by power series in ε whose convergence is *a priori* difficult to ascertain.

As in the case of the Fermi gas, setting $\underline{c}_h = (a_h, b_h)$ the very definition of such constants in terms of sums of infinitely many Feynman graphs implies that they verify a recursion relation

4.15
$$\underline{c}_h = \Lambda \underline{c}_{h+1} + \underline{\mathcal{B}}_h(\underline{c}_{h+1}, \dots, \underline{c}_0), \qquad h \le 0 \tag{4.15}$$

where Λ is a 2×2 diagonal matrix with diagional elements $2^2, 2$, and $\underline{c}_1 \equiv \underline{0}$ because by definition $\Gamma^{(h)} \neq g^{(h)}$ only for $h \le 0$. The analogy with the previous Fermi system problem seems quite strong. The function $\underline{\mathcal{B}}_h(\{\underline{c}\})_{h+1,\dots,0}$ *is not homogeneous in the* $\underline{c}_h$: and this is an important difference with respet to the previous fermionic case.

Clearly if $a_h, b_h \neq 0$ for some h we are "lost" because although, *under the boundedness condition* (4.13), on the running couplings the beta function $\mathcal{B}_h$ is well defined by a convergent series, and although the whole rearranged perturbation series is convergent under the same condition, we shall not be able to prove that the running couplings stay small so that (4.13) is selfconsistent. In fact *both* a_h, b_h are "relevant couplings" (because the elements of the matrix Λ are > 1) and the two data a_1, b_1 (which are 0 for $h = 1$ because, by definition, there are no resonances of scale 1) will need to be carefully tuned so that the renormalization group trajectory a_h, b_h that (4.15) generates from a_1, b_1 is bounded: *there are however no free parameters to adjust in (4.1)!*

The situation is very similar to the one met in the Fermi liquid theory: in that case one solves the problem by showing that the beta function vanishes, at least asymptotically, for the marginal couplings λ_h, δ_h: it remains the relevant coupling ν_h which can be bounded only because we have in that problem the freedom of "adjusting" a free parameter (the chemical potential ν).

In the present case we have two relavant parameters, a_h, b_h, and *no free parameter* in the Hamiltonian: the situation would be hopless *unless* it just happened that the correct initial data for a bounded renormalization group trajectory were precisely the ones that we have, namely $a_1 = b_1 = 0$. This means that one should prove the identity $\mathcal{B}_h(\underline{0}) \equiv \underline{0}$ for all $h \le 0$ so that $\underline{c}_h \equiv \underline{0}$ for all $h \le 1$.

The *vanishing* of the beta function at $\underline{c} = \underline{0}$ was understood in [El96] and it can be seen in various ways, see also [Ga94]. It is however always based on symmetry properties of the model (the choice of the origin of $\underline{\psi}$ plays the role of a "gauge symmetry": the interpretation of the cancellations used by [El96] as a consequence of a gauge symmetry was pointed out and clearly stated in [BGK99], although of course the symmetry is used in the analysis in [El96] and in great detail in [Ga94]).

We are back to a very familiar phenomenon in field theory: a non renormalizable theory becomes asymptotically free and in fact analytic in the parameter ε measuring the strength of the perturbation, because of special symmetries which forbid the exponential growth of the relevant couplings in absence of free parameters in the Lagrangian of the model which could possibly be used to control them.

Concerning the originality of the results obtained with the techniques exposed in this paper the following comments may give an idea of the status of the matter:

(1) The KAM theory presented above is only a reinterpretation of the original proofs [El96], following [Ga94] and [GM96a] see also [BGK98]. However even in classical mechanics the method has generated new results. Other problems of the same type that can be naturally interpreted in terms of renormalization group analysis of suitable quantum fields, see [GM96a], [GM96b], and in the theory of the *separatrix splitting*, see [Ga95], [GGM99].

(2) The results on the theory of the Fermi systems were obtained for the first time by the method described above (including the vanishing of the beta function, [BG90]) and have led to the understanding of several other problems [BM95], [BGM99], [BM99], [Ma97], [Ma98a], [Ma98b], [Ma99].

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